

# Dominant property for the Bel-Robinson tensor and tensor $S$

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## Abstract

The Bel-Robinson tensor contains many nice mathematical properties and its dominant energy condition is desirable for describing the positive gravitational energy. The dominant property is a basic requirement for the quasi-local mass, i.e., in small sphere limit. We claim that there exists another option, a linear combination between the Bel-Robinson tensor  $B$  and tensor  $S$ , which contributes the same dominant property. Moreover, using the 5 Petrov types as the verification, we found that this dominant property justification for the Bel-Robinson tensor can be simplified as examining  $B_{0000} \geq |B_{\alpha\beta 00}|$  and  $B_{0000} \geq |B_{0123}|$ , instead of  $B_{0000} \geq |B_{\alpha\beta\lambda\sigma}|$  for all  $\alpha, \beta, \lambda, \sigma = 0, 1, 2, 3$ .

## 1 Introduction

Gravitational energy cannot be localized at a point since it is forbiddance by the equivalence principle. However, the quasi-local method (i.e., small sphere [1]) can solve out this difficulty. For describing the gravitational energy, Bel and Robinson [2, 3, 4, 5] proposed a tensor that is positive definite and satisfy the dominant property [6]. This dominant property is a relevant requirement for the quasi-local mass. The Bel-Robinson tensor also possesses other nice properties: completely symmetric, traceless and divergence free.

The quasi-local mass has been studied for a long time. There were many people attempted to give a definition for this subject: Harking [7], Penrose [8], Brown and York [9], etc. Recently, Wang and Yau [10] proposed certain requirements and one of them is the Bel-Robinson tensor in vacuum. Indeed, in order to obtain this positivity in small sphere, it is believed that it should be proportional to the Bel-Robinson tensor [11]. However, would it be the only choice? We claim that there exists another option, a linear combination between the Bel-Robinson tensor  $B$  and tensor  $S$ , which contributes the same dominant property (i.e., see (5)).

In principle, because of the symmetry, the Bel-Robinson tensor contains 35 components. Using the 5 Petrov types [12] as the verification, we observe that the dominant property justification for  $B$  can be simplified as examining  $B_{0000} \geq |B_{\alpha\beta 00}|$  and  $B_{0000} \geq |B_{0123}|$ , instead of  $B_{0000} \geq |B_{\alpha\beta\lambda\sigma}|$  for all  $\alpha, \beta, \lambda, \sigma = 0, 1, 2, 3$ . Because all components contain in  $B$  can be written in terms of  $B_{\alpha\beta 00}$  and  $B_{0123}$ .

## 2 Dominant energy condition for $B + sS$

Analogy with the theory of electrodynamics, the Bel-Robinson tensor is defined as

$$B_{\alpha\beta\lambda\sigma} := C_{\alpha\xi\lambda\kappa} C_{\beta}^{\xi}{}_{\sigma}{}^{\kappa} + *C_{\alpha\xi\lambda\kappa} *C_{\beta}^{\xi}{}_{\sigma}{}^{\kappa}, \quad (1)$$

where  $C_{\alpha\beta\mu\nu}$  is the Weyl conformal tensor and its dual  $*C_{\alpha\beta\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta\lambda\sigma} C^{\lambda\sigma}{}_{\mu\nu}$  [13]. As the Weyl tensor and Riemann tensor are equivalent in vacuum, the energy density for this Bel-Robinson tensor becomes

$$B_{\alpha\beta\lambda\sigma} t^{\alpha} t^{\beta} t^{\lambda} t^{\sigma} = E_{ab} E^{ab} + H_{ab} H^{ab}, \quad (2)$$

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which is non-negative and  $t^\alpha$  is the timelike unit normal. Here the Greek letters refer to spacetime and Latin stand for space. In vacuum, the Bel-Robinson tensor and tensor  $S$  [14] can be defined as follows

$$B_{\alpha\beta\lambda\sigma} := R_{\alpha\xi\lambda\kappa}R_{\beta}^{\xi}{}_{\sigma}{}^{\kappa} + R_{\alpha\xi\sigma\kappa}R_{\beta}^{\xi}{}_{\lambda}{}^{\kappa} - \frac{1}{8}g_{\alpha\beta}g_{\lambda\sigma}R_{\rho\tau\mu\nu}^2, \quad (3)$$

$$S_{\alpha\beta\lambda\sigma} := R_{\alpha\lambda\xi\kappa}R_{\beta\sigma}^{\xi\kappa} + R_{\alpha\sigma\xi\kappa}R_{\beta\lambda}^{\xi\kappa} + \frac{1}{4}g_{\alpha\beta}g_{\lambda\sigma}R_{\rho\tau\mu\nu}^2, \quad (4)$$

where  $R_{\rho\tau\mu\nu}^2 = R_{\rho\tau\mu\nu}R^{\rho\tau\mu\nu}$  is the Kretschmann scalar. The symmetric property for  $S$  is  $S_{\alpha\beta\lambda\sigma} = S_{(\alpha\beta)(\lambda\sigma)} = S_{\lambda\sigma\alpha\beta}$ . It is known that the Bel-Robinson tensor possesses the dominant property  $B_{\alpha\beta\lambda\sigma}u^\alpha v^\beta w^\lambda z^\sigma \geq 0$ , where  $u, v, w, z$  are any future-pointing causal vectors. This dominant energy condition is significant for defining the quasi-local mass [10]. Here we propose another option, a linear combination between  $B$  and  $S$  such that it also possesses the dominant property:

$$(B_{\alpha\beta\lambda\sigma} + sS_{\alpha\beta\lambda\sigma})u^\alpha v^\beta w^\lambda z^\sigma \geq 0, \quad (5)$$

where  $s$  is a non-zero small constant.

For the dominant property (i.e., dominant super-energy condition), Senovilla proposed a definition (see Lemma 4.1 of [6]): “If a tensor  $T_{\mu_1\dots\mu_s}$  satisfies the dominant super-energy property, then  $T_{0\dots 0} \geq |T_{\mu_1\dots\mu_s}|$ ,  $\forall \mu_1, \dots, \mu_s = 0, \dots, n-1$  in any orthonormal basis  $\{\vec{e}_\nu\}$ ”. For example, using the 5 Petrov types as the examination, the Bel-Robinson tensor fulfills  $B_{0000} \geq |B_{\alpha\beta\lambda\sigma}|$  requirement. Likewise, for  $B + sS$  and we found there exists a non-zero small  $s$  such that

$$B_{0000} + sS_{0000} \geq |B_{\alpha\beta\lambda\sigma} + sS_{\alpha\beta\lambda\sigma}|. \quad (6)$$

Thus, we suggest that the quasi-local mass should include this extra candidate  $B + sS$  in small sphere. Referring to Szabados’s argument [11], “Therefore, in vacuum in the leading  $r^5$  order any coordinate and Lorentz-covariant quasi-local energy-momentum expression, which is nonspacelike and future pointing must be proportional to the Bel-Robinson ‘momentum’  $B_{\beta\lambda\sigma\alpha}t^\beta t^\lambda t^\sigma$ .” We claim that  $B + sS$  is not only satisfy the causal, but also the dominant property. However, there is a disadvantage for  $B + sS$  because we need to check  $s$  in every physical system. Nevertheless, the advantage for  $B + sS$  gives a relaxation opportunity since obtaining the pure Bel-Robinson tensor for a quasi-local expression is not easy.

Here we consider the total energy-momentum complex which accurate to zeroth order in matter and second order in empty spacetime

$$\mathcal{T}^\alpha{}_\beta = T^\alpha{}_\beta + t^\alpha{}_{\beta\lambda\sigma}x^\lambda x^\sigma, \quad (7)$$

where  $T^\alpha{}_\beta$  is the stress tensor and  $t^\alpha{}_{\beta\lambda\sigma}$  is the gravitational pseudotensor. Note that there are 2 free indices in  $\mathcal{T}_{\alpha\beta}$ . Confining within the small sphere region,  $t^\alpha{}_{\beta\lambda\sigma}x^\lambda x^\sigma$  satisfies the divergence free condition [15]:  $\partial_\alpha(t^\alpha{}_{\beta\lambda\sigma}x^\lambda x^\sigma) = 0$ . The gravitational energy-momentum in small sphere is

$$\int_V t_{\alpha\beta\lambda\sigma}x^\lambda x^\sigma d^3x = \frac{4\pi}{15}(t_{\alpha\beta\lambda\sigma}\eta^{\lambda\sigma} + t_{\alpha\beta 00})r^5, \quad (8)$$

where we used the spherical coordinates and allow the time component be constant for simplicity. Indeed  $\mathcal{T}_{\alpha\beta}$  is symmetric in  $\alpha, \beta$ . Moreover, the dominant energy condition confined in small sphere limit is

$$t_{\alpha\beta 00}u^\alpha v^\beta \geq 0, \quad (9)$$

where  $t_{\alpha\beta\lambda\sigma}\eta^{\lambda\sigma}$  is an arbitrary constant according to the symmetry. If  $t$  is replaced by  $B$  and  $B + sS$  respectively, we have the simplified dominant property representation:

$$B_{\alpha\beta 00}u^\alpha v^\beta \geq 0, \quad (B_{\alpha\beta 00} + sS_{\alpha\beta 00})u^\alpha v^\beta \geq 0. \quad (10)$$

The second inequality is valid for a suitable non-zero small  $s$ .

What is the criterion for selecting the small  $s$ ? Here we give a concrete example by using an isotropic Schwarzschild line element in polar coordinates

$$ds^2 = -(1 - 2Mr^{-1})dt^2 + (1 - 2Mr^{-1})^{-1}dr^2 + r^2(dr^2 + \sin^2\theta d\phi^2), \quad (11)$$

with the assumption that  $M/r \ll 1$ , both the gravitational constant  $G$  and speed of light  $c$  are unity. For simplicity, using the orthonormal basis, there are only three non-vanishing components  $(E_{11}, E_{22}, E_{33}) = (-2, 1, 1)Mr^{-3}$ . The value for the quadratic scalar is  $R^2_{\rho\tau\lambda\sigma} = 48M^2r^{-6}$ . The non-vanishing components for  $B$  and  $S$  are

$$\begin{aligned} (B_{0000}, B_{0011}, B_{0022}, B_{0033}) &= (6, -2, 4, 4)M^2r^{-6}, \\ (B_{1111}, B_{2222}, B_{3333}, B_{1122}, B_{1133}, B_{2233}) &= (6, 6, 6, -4, -4, 2)M^2r^{-6}, \\ (S_{0000}, S_{0011}, S_{0022}, S_{0033}) &= (12, -28, -16, -16)M^2r^{-6}, \\ (S_{0101}, S_{0202}, S_{0303}, S_{1111}, S_{2222}, S_{3333}) &= (8, 2, 2, 12, 12, 12)M^2r^{-6}, \\ (S_{1122}, S_{1133}, S_{2233}, S_{1212}, S_{1313}, S_{2323}) &= (16, 16, 28, -2, -2, -8)M^2r^{-6}. \end{aligned} \quad (12)$$

Obviously, the Bel-Robinson tensor fulfills the dominant energy condition. Similarly, we find that  $(B + sS)$  satisfies the dominant property requires  $s \in [-\frac{1}{14}, \frac{1}{4}]$ . In particular, for the Landau-Lifschitz (LL) pseudo-tensor [16], evaluated in the Riemann normal coordinates, satisfies the dominant property:

$$\partial_{\mu\nu}^2 t_{LL}^{\alpha\beta} = \frac{1}{9} \left( 7B^{\alpha\beta}_{\mu\nu} + \frac{1}{2}S^{\alpha\beta}_{\mu\nu} \right). \quad (13)$$

### 3 Conclusion

The Bel-Robinson tensor  $B$  has the dominant energy condition and this is a requirement for describing the quasi-local mass in small sphere. We discovered that there exists an opportunity tensor  $B + sS$  such that this combination also contributes the same dominant property. As it is not easy for achieving a multiple of the pure Bel-Robinson tensor in quasi-local expression, then  $B + sS$  provides a relaxation opportunity for the dominant energy condition. Moreover, we also pointed out that the examination for the dominant property can be simplified for the Bel-Robinson tensor. Using the 5 Petrov types, instead of verifying  $B_{0000} \geq |B_{\alpha\beta\lambda\sigma}|$  for all  $\alpha, \beta, \lambda, \sigma = 0, 1, 2, 3$ , it is enough to check  $B_{0000} \geq |B_{00\alpha\beta}|$  and  $B_{0000} \geq |B_{0123}|$ .

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